

GLOBALLY HYPERBOLIC SPACETIMES WITH SINGULARITIES

CRISTI STOICA

ABSTRACT. Black hole singularities are usually considered to break the time evolution, in (classical) General Relativity. It is shown here that, in fact, the standard black hole solutions are compatible with global hyperbolicity. For this, globally hyperbolic spacetimes containing singularities are constructed.

The first step is to extend solutions of the Einstein equation to the singularities, in a way which avoids infinities. The stationary black hole solutions are then analytically extended beyond the singularities. Next, the topology of the spacetime at the singularity is repaired, by using the new extensions. Then, solutions are constructed, which are shown to be globally hyperbolic by foliating the spacetime with Cauchy hypersurfaces. The foliations are constructed by applying a Schwarz-Christoffel mapping to the Penrose-Carter representations of the black hole solutions.

The results are applied to non-stationary black holes, including evaporating ones.

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Email: holotronix@gmail.com

*“I was borne violently into the channel of the Ström,
and in a few minutes, was hurried down the coast into
the ‘grounds’ of the fishermen.”*

Edgar Allan Poe, A Descent into the Maelström, 1841

INTRODUCTION

0.1. The singularity theorems. Despite the successes of General Relativity, one of its own consequences seems to question it: the occurrence of singularities in the black holes. It is often said that General Relativity predicts, because of these singularities, it’s own breakdown [11, 9, 1, 12, 2, 3]. Such singularities follow from the *singularity theorems* of Penrose and Hawking [21, 7, 11, 10]. The conditions leading to singularities were found to be common (Christodoulou [4]), and then even more common (Klainerman and Rodnianski [15]).

Initially, there was some confusion regarding the singularities. In 1916, when Schwarzschild proposed [25, 24] his solution to Einstein’s equation, representing a black hole, it was believed that the event horizon is singular. Only in 1924, when Eddington proposed another coordinate system which removed the singularity at the event horizon, it was understood that it was only apparent, being due to the choice of the coordinate system. But the singularity at the center of the black hole remained independent of the particular coordinates, and the singularity theorems showed that any black hole would have such a singularity. In [29, 30, 31] we have shown that, although the genuine singularities cannot be removed, they can at least be made manageable – there are coordinate changes which make the metric degenerate, but smooth.

0.2. The black hole information paradox. Soon (in its proper time) after an object passes through the event horizon, it reaches the singularity of the black hole. All the information contained in it seems to vanish in the singularity.

On the other hand, the equations governing the physical laws are in general reversible, guaranteeing that no information can be lost. But according to Hawking [8, 9] the black hole may emit radiation and evaporate. If the black hole evaporates completely, it seems to leave behind no trace of the information it swallowed. Moreover, it seems to be possible for an originally pure state to end up being mixed, because the density matrix of the particles in the black hole’s exterior is obtained by tracing over the particles lost in the black hole with which they were entangled. This means that the unitarity appears to be violated, and the problem becomes even more acute.

0.3. The meaning of singularities. The singularities in General Relativity are places where the evolution equations cannot work, because the involved fields become infinite. If the Cauchy surface on which the fields are defined is affected by singularities, then the equations cannot be developed in time.

From geometric viewpoint, these singularities are points where the metric becomes singular, and the geodesics become *incomplete*. Since we don't know to extend the fields to such points, normally we remove them from the spacetime.

Actually, we can rewrite the fields involved, and the equations defining them, so that the fields remain finite at any point [6, 26, 29, 30, 31]. At the points where the metric is non-singular, the equations remain equivalent to Einstein's equation.

With this modification, the spacetime can be extended to the singular points.

Once we have the fields and the topology repaired, we have to check that we can choose a maximal globally hyperbolic spacetime (or equivalently, admitting a Cauchy foliation) so that the evolution equations can be defined.

Consequently, a natural interpretation of the singularities emerges, which makes them harmless for the physical law, in particular for the information conservation.

We will illustrate this approach on the black hole solutions known from the literature.

1. CANCELING THE SINGULARITIES OF THE FIELD EQUATIONS

Some of the tensor fields involved in Einstein's field equation become infinite at the singularity points. We need a method to replace them with other fields which obey equations equivalent to Einstein's at the non-singular points, but remain in the same time smooth at the singularity. Our proposed approach generalizes the semi-Riemannian geometry, so that it works at an important class of singularity points [26, 27, 28].

Other method, used by Einstein and Rosen (for which they credited Mayer [6]) is to multiply the Einstein equations with by a well-chosen power of $\det g$. This way, all instances of g^{ab} in the expression of $\det g^2 \text{Ric}$ are replaced by $\det g g^{ab}$, the adjugate matrix of g_{ab} ([6], p. 74), and the singularities of the Riemann and the Ricci tensors can be canceled out for certain cases. The resulting equations still make sense as partial differential equations in the components of the metric tensor. Of course, the invariants of Semi-Riemannian Geometry have geometric

meanings so long as the metric tensor is non-singular. But once we allow it to become singular, the invariants like covariant derivative and curvature become undefined. The quantities which replace them if we multiply the equations with $\det g$ or other factors cannot give them this meaning. Maybe a new kind of geometry is required to restore the ideas of covariant derivative and curvature for singular metrics.

We would prefer to have a generalization of Semi-Riemannian Geometry, which allows us to define geometric objects like covariant derivative, Riemann curvature and Ricci tensor even when the metric becomes degenerate.

We did a modest step in this direction, by developing a Singular Semi-Riemannian Geometry for a class of metrics which can change the signature and become degenerate [26, 27, 28]¹.

Our extension of Semi-Riemannian Geometry allows one to define in some cases covariant derivative, the Riemann tensor R_{abcd} (but not R^a_{bcd}), the Ricci tensor and the scalar curvature (hence the Einstein tensor). It can be used to write Einstein's equation, which in general becomes singular when the signature changes, but we can replace it with a densitized version which removes its singularities.

The metric of a stationary black hole solution like the Schwarzschild, Reissner-Nordström, and in general Kerr-Newman, has components which become infinite at singularity. Since the methods we developed in [26, 27, 28] work for metrics whose singular character manifests only by being degenerate, we may think that the stationary black hole solutions cannot be covered. But, in fact, we can make the singular metric be of degenerate type, by applying certain coordinate changes. We did this for the Schwarzschild metric in [29], for the Reissner-Nordström metric in [30], and for the Kerr-Newman metric in [31].

This approach is similar to the Eddington-Finkelstein coordinates, which were used to remove the apparent singularity on the event horizon of the Schwarzschild black hole. The new coordinates were chosen so that they are themselves singular at the (apparent) singularity, canceling it. In the case of the Eddington-Finkelstein coordinates, the resulting metric is non-degenerate on the event horizons, where previously appeared to be singular. In the case of the genuine singularities of the stationary black hole solutions, the metric obtained by the new coordinates we propose has analytic components, and its singularity reduces to the fact that it is degenerate.

¹Important results for the case when the metric has constant signature were been previously developed by D. Kupeli in [16, 17], by a different approach. Previous results were obtained by Moisil [18], Strubecker [32, 33, 34, 35], and Vrănceanu [36].

In the following, we need to explore the arena on which are defined the fields – the spacetime – to make sure that the singularities don't alter the topology in a way which breaks down the evolution equations.

2. THE TOPOLOGY OF SINGULARITIES

In the following we shall see that the main black hole solutions of Einstein's equations can be interpreted so that the time evolution is not jeopardized. Two main factors help us to do this:

- (1) Remove the singularities from the field equations.
- (2) Make sure that the time evolution takes place on a space whose topology doesn't change in time.

In the current section, we shall see that these conditions can be ensured for the typical black hole solutions of Einstein's equation.

2.1. The final space-like singularity from Schwarzschild's solution. The Schwarzschild solution has the particularity that its singularity is spacelike. Although in some coordinate systems (like the original Schwarzschild coordinates) the singularity is apparently timelike and one dimensional, it is in fact spacelike, as we can see by choosing Kruskal-Szekeres coordinates or Penrose-Carter coordinates.

Let's take the Penrose-Carter diagram of a Schwarzschild black hole (Figure 1).

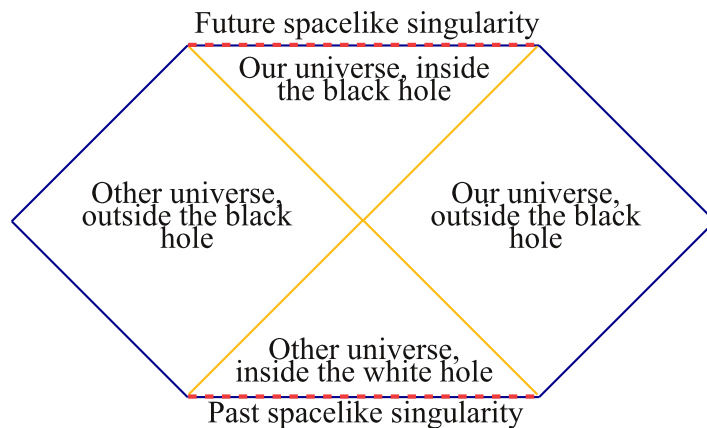


FIGURE 1. The maximally extended Schwarzschild solution, in Penrose-Carter coordinates.

This diagram actually represents the maximally extended Schwarzschild solution, in Penrose-Carter coordinates. This extended solution is interpreted to include, together with the universe containing the black hole, another universe, in which there is a white hole.

We can foliate this spacetime with Cauchy hypersurfaces, as we can see in Figure 2.

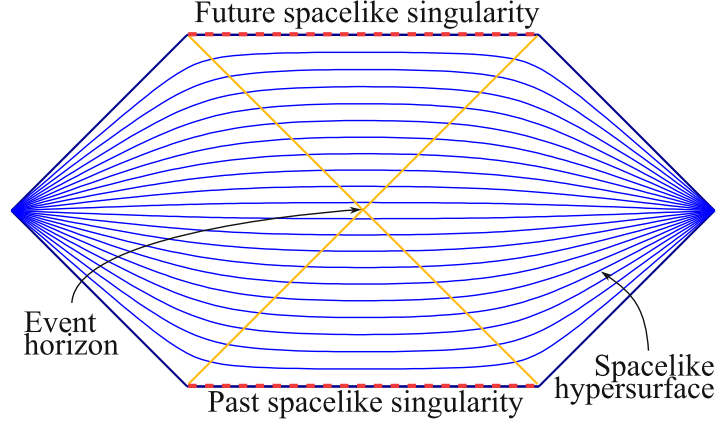


FIGURE 2. Space-like foliation of the maximally extended Schwarzschild solution.

This foliation is obtained with the help of the Schwarz-Christoffel mapping. There is a version of the Schwarz-Christoffel mapping which maps the strip

$$(1) \quad \mathcal{S} := \{z \in \mathbb{C} | \text{Im}(z) \in [0, 1]\}$$

to a polygonal region from \mathbb{C} , with the help of the formula

$$(2) \quad f(z) = A + C \int^{\mathcal{S}} \exp \left[\frac{\pi}{2} (\alpha_- - \alpha_+) \zeta \right] \prod_{k=1}^n \left[\sinh \frac{\pi}{2} (\zeta - z_k) \right]^{\alpha_k - 1} d\zeta,$$

where $z_k \in \partial \mathcal{S} := \mathbb{R} \times \{0, i\}$ are the prevertices of the polygon, and $\alpha_-, \alpha_+, \alpha_k$ are the measures of the angles of the polygon, divided by π (cf. e.g. [5]). The vertices having the angles α_- and α_+ have as prevertices the ends of the strip, which are at infinite. The foliation is given by the level curves $\{\text{Im}(z) = \text{const.}\}$. In this article, all polygons to which we will apply the Schwarz-Christoffel mapping have a common property. They all have $\alpha_- = \alpha_+$ and the edges inclined at most at $\frac{\pi}{4}$, alternating in such a way that the level curves with $\text{Im}(z) \in (0, 1)$ have at each point tangents making an angle strictly between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$. This ensures that our foliations are indeed spacelike.

To obtain the foliation from Figure 2, we take the prevertices to be

$$(3) \quad (-\infty, -a, a, +\infty, a + i, -a + i),$$

where $a > 0$ is a real number. The angles are respectively

$$(4) \quad \left(\frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{4} \right).$$

This foliation contains the past white hole, which one may consider unphysical. We can make a similar foliation, this time without the white hole, if we use the prevertices

$$(5) \quad (-\infty, -a, 0, a, +\infty, b+i, -b+i),$$

where $0 < b < a$ are positive real numbers. The angles are respectively

$$(6) \quad \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{4} \right).$$

The resulted foliation is represented in Figure 3.

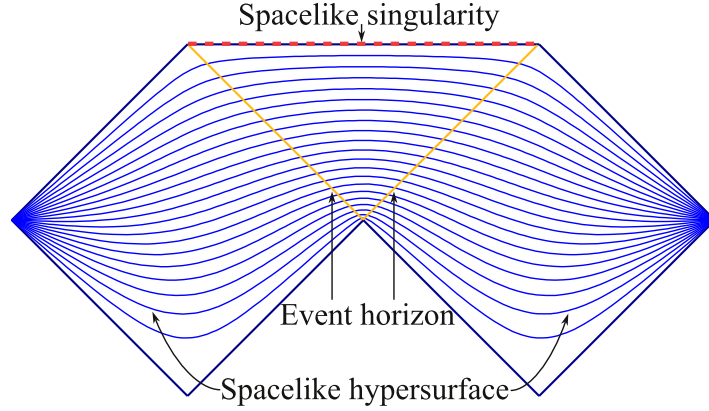


FIGURE 3. Space-like foliation of the Schwarzschild solution, without white hole.

2.2. Timelike singularities of charged black holes. The stationary solutions of Einstein's equations representing charged black holes were discovered by Reissner [23] and Nordström [20]. The Penrose-Carter diagrams of the maximally extended solutions are represented in Figure 4 A for naked black holes, 4 B for extremal black holes, and 4 C for non-extremal black holes.

In [30] we proposed new coordinates for the Reissner-Nordström metric, which make the metric smooth at the singularity. The singularity appears to be on a timelike line, with respect to which the solution is symmetric. We take as prevertices of the Schwarz-Christoffel mapping (2) the set

$$(7) \quad (-\infty, -a, 0, a, +\infty, i),$$

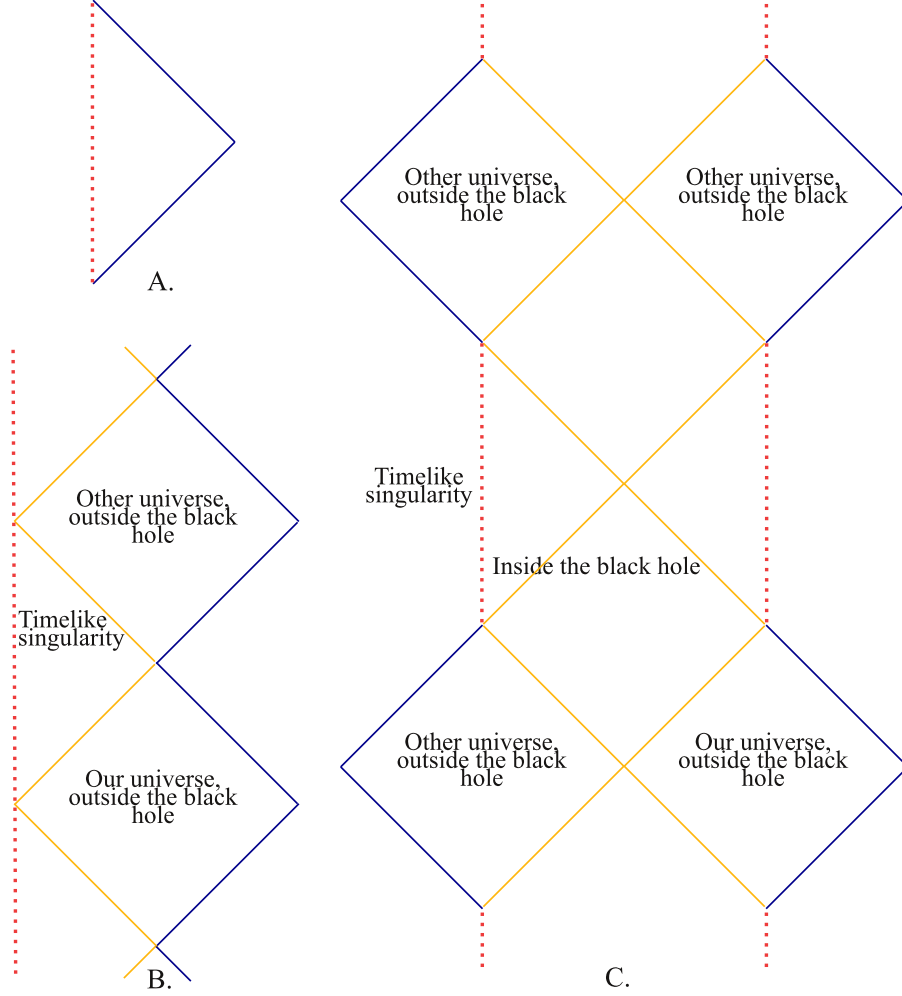


FIGURE 4. **A.** Naked Reissner-Nordström black holes. **B.** Extremal Reissner-Nordström black holes. **C.** Non-extremal Reissner-Nordström black holes.

where $0 < a$ is a positive real number. The angles are respectively

$$(8) \quad \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} \right).$$

By an appropriate choices of a we get the foliation represented in diagrams 5, 6, respectively 7, corresponding to the Figures 4 A, 4 B, respectively 4 C.

These solutions ignore the other universes appearing when the equations are analytically extended to their maximum domain in Penrose-Carter coordinates. It is debatable if we really need, from physical

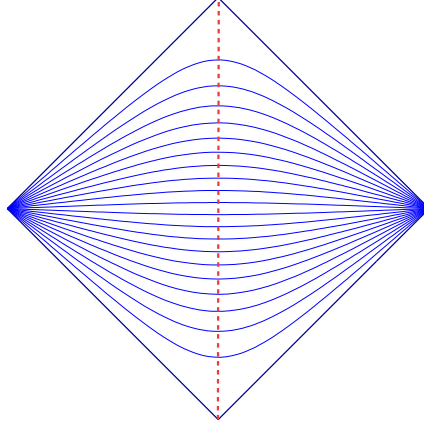


FIGURE 5. Space-like foliation of the naked Reissner-Nordström solution.

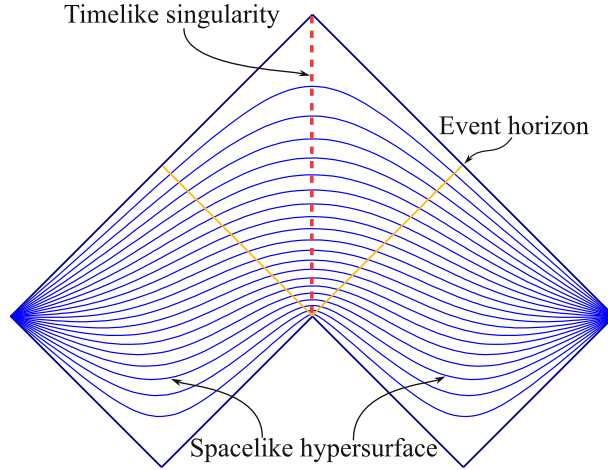


FIGURE 6. Space-like foliation of the extremal Reissner-Nordström solution.

viewpoint, to include extensions beyond the Cauchy horizons, and by them the infinite number of other universes appearing in such maximal extensions. Our foliation takes the outside and the inside of the black hole and foliates it as required to solve this problem. Taking the maximally extended solution would not allow us to do this, because it contains Cauchy horizons, beyond which the evolution equations cannot be evolved causally ([10], p. 159, [37], p. 317).

Also, we shall see that, by limiting to one universe in these solutions, instead of taking the maximal extensions, we can combine them with non-singular solutions and have black holes with a beginning and an

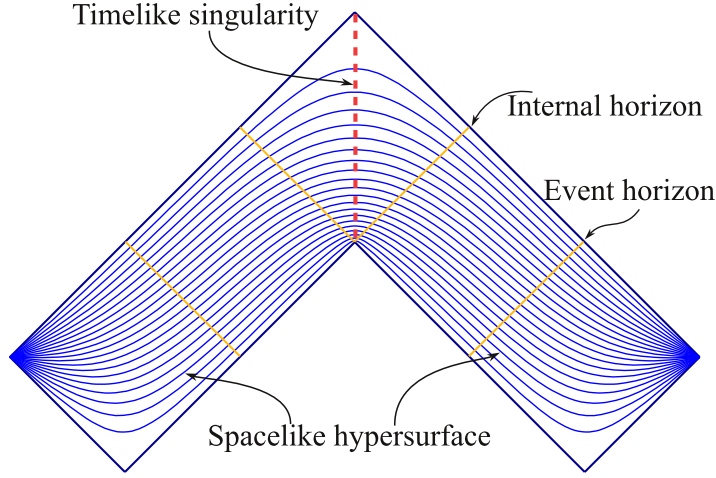


FIGURE 7. Space-like foliation of the Reissner-Nordström solution.

end. Such black holes seem more plausible from physical viewpoint, although we will not exclude the primordial and eternal black holes.

2.3. Timelike singularities of rotating black holes. The rotating neutral black holes (Kerr, [13]), and the rotating electrically charged black holes (Kerr-Newman, [19]) don't have spherical symmetry, they are axisymmetric. Their Penrose-Carter diagrams are similar with those for Reissner-Nordström black holes (Figure 4), in coordinates (t, r) , but there are some major differences. In Boyer-Lindquist coordinates the singularity appears to be a ring, and in its neighborhood the coordinate φ becomes timelike – the curves defined by $(t = \text{const.}, r = \text{const.}, \vartheta = \text{const.})$ are closed timelike curves (*cf. e.g.* [10], §5.6, [37], p. 315)². In [31] we proposed new coordinates, which make the metric analytic at the singularities, and remove, as a byproduct, the closed timelike curves. These new extensions of the metric can be taken so that the solution looks more like the Reissner-Nordström solution. So, the Penrose-Carter diagrams for our extensions of the Kerr and Kerr-Newman solutions look very similar to those for the Reissner-Nordström solution (Fig. 8).

²It has been proposed that the closed time-like curves depend on the particular coordinate system used [14].

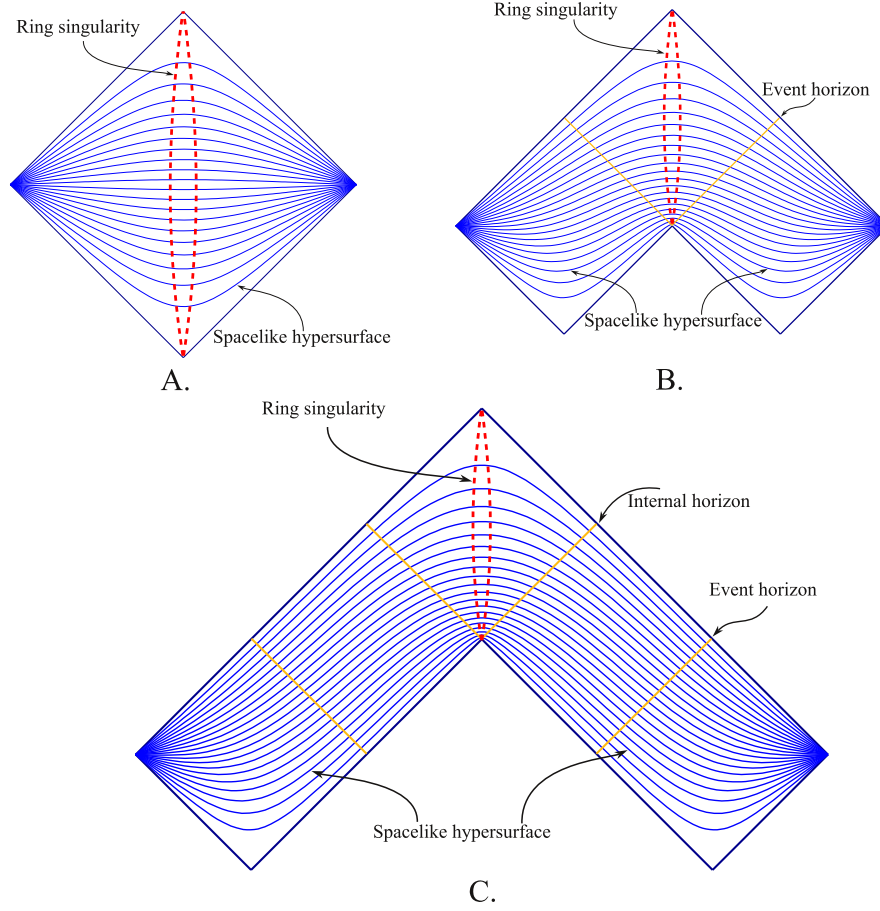


FIGURE 8. **A.** Space-like foliation of the naked Kerr-Newman solution ($a^2 + q^2 > m^2$). **B.** Space-like foliation of the extremal Kerr-Newman solution with $a^2 + q^2 = m^2$. **C.** Space-like foliation of the non-extremal Kerr-Newman solution ($a^2 + q^2 < m^2$).

3. NON-STATIONARY BLACK HOLES

The stationary solutions are idealizations. In reality, a black hole doesn't necessarily exist from the beginning of the universe, it may be younger. Also, it may evaporate completely after a finite time interval.

For example, a spacelike foliation of a non-rotating and electrically neutral black hole which formed after the beginning of the Universe, and which continues to exist forever, is represented in Figure 9.

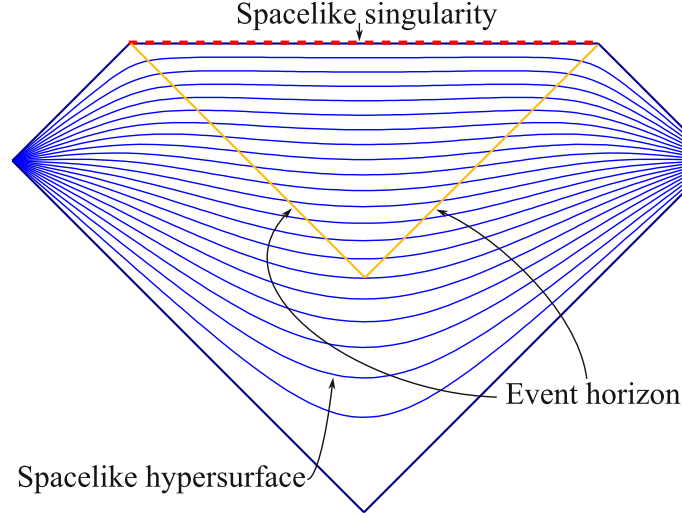


FIGURE 9. The spacelike foliation of a non-rotating and electrically neutral black hole formed after the beginning of the Universe, which continues to exist forever.

The prevertices of the Schwarz-Christoffel mapping (2) whose image is represented in Figure 9 are given by the set

$$(9) \quad (-\infty, 0, +\infty, a + i, -a + i),$$

where $0 < a$ is a positive real number. The angles are respectively

$$(10) \quad \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{4} \right).$$

We see that, because the typical spacelike hypersurface in the foliation in Figure 3 is diffeomorphic with the space \mathbb{R}^3 of the Minkowski spacetime $\mathbb{R}^3 \times \mathbb{R}$, the topology doesn't change because of the occurrence of a neutral non-rotating black hole. In fact, the typical spacelike hypersurface of the foliation of a charged and rotating black hole also has the same topology as \mathbb{R}^3 , as we have seen. So they can appear and evaporate as well in an \mathbb{R}^3 space, without disrupting the topology. This condition is required to have a good time evolution.

If the non-rotating and electrically neutral black hole is primordial (exists from the beginning of the universe), but evaporates completely after a finite time, the spacelike foliation is as represented in Figure 10. The prevertices and the angles are identical to those for Figure 6.

If this non-rotating and electrically neutral black hole is not primordial and it evaporates completely after a finite time, then at large distances the spacetime is very close to the Minkowski spacetime, being

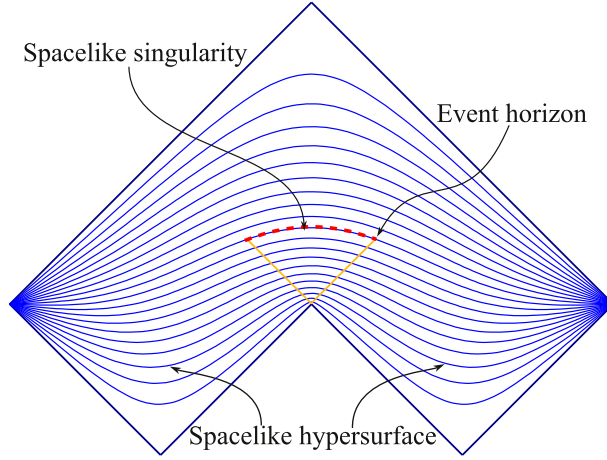


FIGURE 10. The spacelike foliation for a non-rotating and electrically neutral primordial black hole, which evaporates after a finite time.

asymptotically flat. Consequently, a spacelike foliation looks like that in Figure 11.

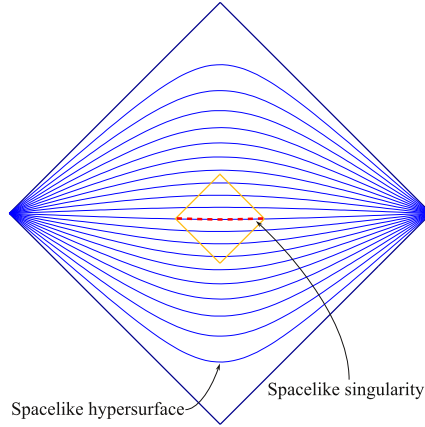


FIGURE 11. The spacelike foliation for a non-rotating and electrically neutral black hole formed after the beginning of the Universe, and which evaporates after a finite time.

The prevertices of the diagram represented in Figure 11 are

$$(11) \quad (-\infty, 0, +\infty, i),$$

and the angles are

$$(12) \quad \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} \right).$$

Let's see now what happens if the singularity is timelike, as in the case of the charged and/or rotating black holes. If the black hole is primordial and evaporates, the corresponding spacelike foliation is represented in Figure 12.

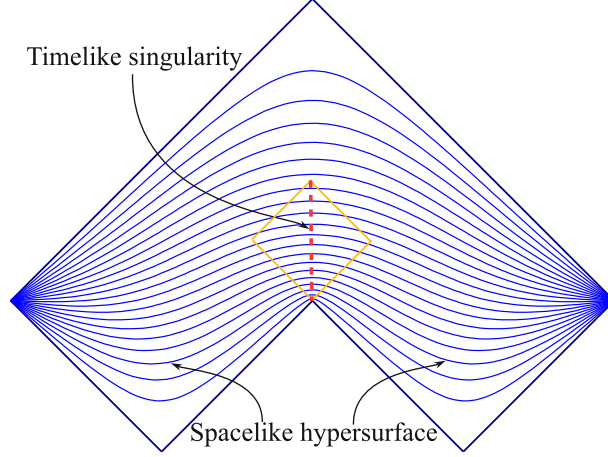


FIGURE 12. Primordial evaporating black hole with timelike singularity.

The prevertices and the angles are the same as those for the Figure 6.

The spacelike foliation of a spacetime containing a black hole which is not primordial and does not evaporate is represented in Figure 13. The prevertices and the angles are again those from equations (11) and (12).

The same prevertices and angles are used to construct the spacelike foliation for a non-primordial evaporating black hole, represented in Figure 14. If the black hole rotates, the singularity is ring-shaped (as in Fig. 8), but the diagrams are similar.

4. TIME EVOLUTION IN SPACETIMES WITH SINGULARITIES

The conformal structure is given equivalently by the lightcones. This structure is involved in the Cauchy development in General Relativity. Namely, each point in spacetime is affected by what happens in its past lightcone.

In our formulation the singularities are repaired by a coordinate change, and the topology of the spacelike hypersurface is preserved by the time evolution. The conformal structure is identical, in all cases examined, to the conformal structure of non-singular spacetimes which

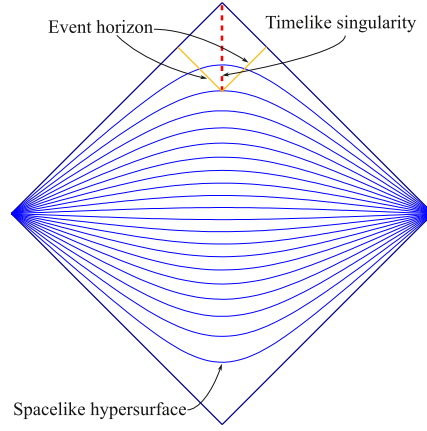


FIGURE 13. Non-primordial non-evaporating black hole with timelike singularity.

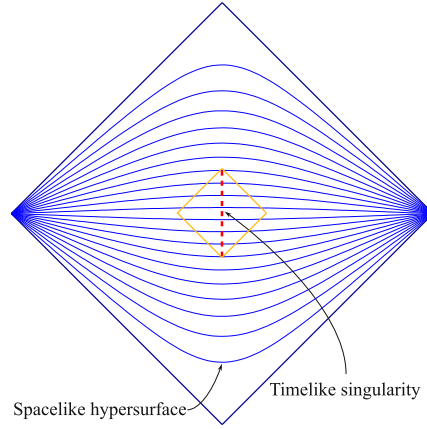


FIGURE 14. Non-primordial evaporating black hole with timelike singularity.

are globally hyperbolic. Namely, their Penrose-Carter diagrams are all conformally equivalent to that of the Minkowski spacetime. This is evident from the conformal equivalence of all solutions with the strip \mathcal{S} from (1), via a Schwarz-Christoffel mapping of the form (2). Therefore, the property of global hyperbolicity of the Minkowski spacetime transfers to our solutions as well.

This analysis shows that, even in the presence of singularities, the physical laws don't necessarily break down. We can make

- (1) an appropriate modification of the equations so that the metric becomes smooth at the singularities (see §1)

- (2) an appropriate foliation of the spacetime into spacelike hypersurfaces (*cf.* §2 and §3)
- (3) an appropriate extension of the spacetime at singularities, so that the topology of the spacelike hypersurfaces of the foliations is preserved (*cf.* §2 and §3).

These choices, which are not determined by the non-singular General Relativity, ensure us that the time evolution is not disrupted, and the Cauchy data, hence the information, is preserved. Also, this allows the unitarity to be restored, because there is no loss in the information, and a pure state can no longer become mixed by this mechanism.

5. SINGULAR GENERAL RELATIVITY

There are many actions we can take when encountering singularities in the solutions of Einstein's equation. There is no need to *cancel* them [22], or to say about them “here be dragons” – we just have to explore them theoretically.

This article presents an approach to the problems posed by the singularities.

One way to repair the singularities in the Einstein equation is by using the idea proposed by Einstein and Rosen [6] (see §1).

But we can do even more. Einstein's equation is an identity between Einstein's tensor, which is defined by the curvature, and the stress-energy tensor of the matter fields. By multiplying them with some functions, so that the partial differential equations expressed in a coordinate system become non-singular, we can indeed extend them at the singularities. But the quantities involved now seem to have no meaning, unlike the original terms.

This suggests that it may be useful to develop the Semi-Riemannian Geometry to deal with such situations when the metric is degenerate or singular.

We already explored the case when the metric is allowed to pass from being non-degenerate to being degenerate in [26, 27, 28]. We showed there that we can still have invariants like the familiar Riemann curvature and like the covariant derivative, if the metric satisfies some reasonable conditions. This allowed us to write a densitized version of Einstein's equation, which remains smooth at the singularities due to the degeneracy of the metric, although the curvature operator $R^a{}_{bcd}$, and the Kretschmann scalar $R_{abcd}R^{abcd}$ become singular. Normally, the Einstein tensor $R_{ab} - \frac{1}{2}g^{ab}R_{ab}$ also becomes singular, in which case we

can work with its densitized version. But there are cases when it remains smooth, for example in the vacuum solutions like Schwarzschild's and Kerr's (when it is actually 0).

6. CONCLUSIONS

This article shows how one can extend smoothly the black hole solutions at the singularities, then find an appropriate foliation of the spacetime into spacelike hypersurfaces, and an appropriate extension of the spacetime at singularities, so that the topology of the spacelike hypersurfaces of the foliations is preserved. In our typical examples, the time evolution in the presence of singularities is restored. The examples given here showed this explicitly for the neutral and charged, rotating or non-rotating, primordial or not, evaporating or not black holes. These models have been shown here to have Penrose-Carter diagrams conformally equivalent to that of the Minkowski spacetime, inheriting therefore from the latter the global hyperbolicity. Consequently, the Cauchy data is preserved, and the information loss is avoided. This allows the construction of Quantum Field Theories in such curved spacetimes ([12], p. 9), and the unitarity is restored.

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